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New Results in Signal Design For the AWGN Channel

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13. ABSTRACT (*Maximum 200 words*)

The design of M average power constrained signals in white Gaussian noise (AWGN) is addressed. The long-standing strong simplex conjecture which postulates that the optimal signals, under an average power constraint, are the vertices of a regular simplex, is disproven. A signal set is shown that performs better than the regular simplex at low signal-to-noise ratios for all $M \geq 7$. This leads to a theorem stating that for any fixed $M \geq 7$ no signal set exists that is optimal at all signal-to-noise ratios. Furthermore it is found that the optimal solution at low signal-to-noise ratios is not an equal energy solution for any $M \geq 7$. The regular simplex is shown to be the unique polytope that maximizes the minimum distance between signals. This result leads to the corollary that a signal set which maximizes the minimum distance is not necessarily optimum. However, the regular simplex is shown to be globally optimum in the sense of uniquely maximizing the union bound at all signal-to-noise ratios.

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CONTENTS

NEW RESULTS IN SIGNAL DESIGN FOR THE AWGN CHANNEL

1. INTRODUCTION

The design of efficient signal sets for transmission over channels that are contaminated by Gaussian noise has been an active area of research for many years. A signal set that is more efficient than another will typically result in comparable savings in transmitted energy. Hence, the determination of optimal signal sets is an important problem from a practical communication perspective as well as from a theoretical standpoint. A fair amount of work has been done in the area of signal design. Unfortunately, few results exist on the optimality of signal sets (throughout the paper an optimal signal set is one that maximizes the average probability of detection.) The optimal selection of M-signal vectors embedded in even the most fundamental type of noise, white Gaussian noise, generally is not known. One of the most famous conjectures of communication, dating back to 1948, states that the optimal signal vectors are vertices of an n dimensional regular simplex for which $M = n + 1$[1, p. 74]. When the signal vectors are constrained only by an average power limitation, this conjecture is referred to as the strong simplex conjecture (SSC) [2]. To avoid confusion, we refer to the conjecture of simplex optimality when the signal vectors lie on the surface of a sphere, as the weak simplex conjecture (WSC). The validity of the SSC implies the validity of the WSC, although the converse statement is not true. Note that, irrespective of whether or not the signal vectors are average power constrained or peak power constrained, the capacity of the additive Gaussian noise channel (AGNC) is the same[3][4, p. 324]. Under the assumption that the signal vectors are of equal energy, Balakrishnan proved, in his seminal work [5], that the regular simplex is 1) optimal (in terms of maximizing the average probability of detection) as the signal-to-noise ratio (SNR) λ approaches infinity, 2) optimal as λ approaches zero, and 3) locally optimal at all λ. He also proved that if a signal set *does* exist that is optimal at all λ, it is necessary the regular simplex signal set. In 1967, Dunbridge [6] [7] extended Balakrishnan's work where only an average power constraint is imposed on the signal set. Dunbridge proved that the regular simplex is 1) the optimal signal set as λ approaches infinity and 2) a local extremum at all λ. He also proved that the regular simplex must be the optimal signal set, if one does exist. For the case of

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$M = 2$, the regular simplex or antipodal signal set has been proven to be optimal at all SNR for both the average and peak power constrained channels. Dunbridge proved, under an average power constraint, that the regular simplex with $M = 3$ is optimal as λ approaches zero. Work on the weak simplex conjecture [8] was later shown by Farber [9] to prove the conjecture for $M < 5$. It was proven by both Balakrishnan [10] and Weber [11, p. 215] that the regular simplex maximizes the minimum distance under a peak power constraint.

Generally, the optimal design of signals for the noncoherent Gaussian channel is also unknown. It has been long conjectured that the orthogonal signal set is the global optimum. Weber [11, p. 269] proved that the orthogonal signal set is locally optimum under no bandwidth constraint. Stone and Weber [12] further proved the optimality of the orthogonal signal set as λ approaches infinity. The orthogonal signal set has been proven to be the global optimum for all SNR for the case of $M = 2$ [13]. Cases of restricted dimensionality have been explored [14] for two dimensions and $M = 2, 3, 4, 6, 12$.

A number of new results are presented in this paper. The major result that is presented in Section 3 is a counterexample to the strong simplex conjecture. An explicit signal set is shown to be better than the regular simplex for all $M \geq 7$ under an average power constraint. This leads to a proof establishing, for any $M \geq 7$, that there are no signal sets that are optimal at all signal-to-noise ratios. In Section 4 we prove that the regular simplex uniquely maximizes the minimum distance under an average power constraint. A simple proof that the regular simplex maximizes the minimum distance under a peak power constraint is also shown. This work leads to the corollary that a signal set that maximizes the minimum distance between signals is not necessarily optimum. This is an interesting result, since much signal-design work has been based on maximizing the minimum distance between signals because of the inherent simplicity of the criteria. In Section 5 we address the global optimality of the regular simplex under the performance measure of the union bound on the probability of detection. The union bound is often used to assess the performance of signal sets at medium to high SNR when computation of the probability of detection is intractable. It is proven that the regular simplex uniquely maximizes the union bound at all SNR. The optimality of signal sets at low signal-to-noise ratios is also examined. It is proven that the optimal solution at low signal-to-noise ratios is not an equal energy solution for all $M \geq 7$. Additionally the signal set presented in Section 3 is shown to be the optimal signal set when restricted to all 1-D signal sets.

2. PRELIMINARIES

Consider the transmission of one of M signals $s_i(t)$, $i = 1, \cdots, M$ through a channel contaminated with white Gaussian noise $n(t)$. The signals can be represented through a discrete time n dimensional vector representation $\mathbf{s}_i$ (the time discretization of a continuous time signal is discussed at length in Ref.[11, p. 127]). After transmission we receive the n dimensional vector

$$\mathbf{y} = \mathbf{s}_i + \mathbf{n} \quad i = 1, \cdots, M \tag{1}$$

and wish to determine which of the M signals was transmitted. We define the SNR parameter

$$\lambda^2 = \frac{1}{M} \sum_{i=1}^{M} \mathbf{s}_i^T \mathbf{s}_i \tag{2}$$

where T denotes transpose. It is assumed that $\mathbf{n}$ is a zero mean Gaussian vector with covariance matrix equal to the n by n identity matrix. We define the normalized M dimensional matrix of inner products $\alpha = (\lambda_{ij})$ by,

$$\lambda_{ij} = \frac{\mathbf{s}_i^T \mathbf{s}_j}{\lambda^2}.$$

It immediately follows that

$$\sum_{i=1}^{M} \lambda_{ii} = M. \tag{3}$$

For an equal energy signal set, the λ_{ii}, $i = 1, \cdots, M$ are identical.

The optimum detector, in terms of maximizing the average probability of detection, chooses that signal $\mathbf{s}_i$, which maximizes $p(\mathbf{y}|\mathbf{s}_i)p_i$, where the *a priori* probability of the ith signal is p_i and $p(\mathbf{y}|\mathbf{s}_i)$ is the probability density function of $\mathbf{y}$ conditioned on $\mathbf{s}_i$. The results derived throughout the paper will assume that the signals are equally likely. i.e., $p_i = \frac{1}{M}$. In case of ties, we choose the signal with the smallest index from the $\mathbf{s}_i$ that tied. We can write $p(\mathbf{y}|\mathbf{s}_i)p_i$ as,

$$p(\mathbf{y}|\mathbf{s}_i)p_i = \frac{1}{(2\pi)^{n/2}} \exp[-\frac{1}{2}(\mathbf{y} - \mathbf{s}_i)^T(\mathbf{y} - \mathbf{s}_i) + \log p_i]. \tag{4}$$

Upon defining $E_i(\mathbf{y}) = \mathbf{y}^T\mathbf{s}_i - \frac{\mathbf{s}_i^T\mathbf{s}_i}{2} + \log p_i$, we see that Eq. (4) is maximized when we choose the signal corresponding to the maximum of $E_i(\mathbf{y})$. The average probability of detection is,

$$\begin{aligned} P_d &= \sum_i p_i P(E_i(\mathbf{y}) = \max_j E_j(\mathbf{y})|\mathbf{s}_i) \\ &= \sum_i p_i \int \cdots \int_{\Lambda_i} p(\mathbf{y}|\mathbf{s}_i) dy_1 \cdots dy_n \\ &= \sum_i \frac{1}{(2\pi)^{n/2}} \int \cdots \int_{\Lambda i} \exp[-\frac{1}{2}\mathbf{y}^T\mathbf{y} + E_i(\mathbf{y})] dy_1 \cdots dy_n, \end{aligned}$$

where Λ_i corresponds to the decision region $\{\mathbf{y}|E_i(\mathbf{y}) = \max_j E_j(\mathbf{y})\}$ for cases where a unique maximum of $E_j(\mathbf{y})$ for $j = 1, \cdots, M$ exists. When a unique maximum does not exist, we resolve ties by choosing the signal index from the smallest index among those regions $\{\mathbf{y}|E_i(\mathbf{y}) = \max_j E_j(\mathbf{y})\}$ that overlap. When defined in this manner the Λ_i are disjoint, and the above can be rewritten as

$$P_d = \frac{1}{(2\pi)^{n/2}} \int_{R_n} \exp[-\frac{1}{2}\mathbf{y}^T\mathbf{y} + \max_i E_i(\mathbf{y})] dy_1 \cdots dy_n, \tag{5}$$

where R_n represents n-dimensional Euclidean space. This equation can be examined by defining a n-dimensional Gaussian vector $\mathbf{z}$ with zero mean and identity covariance matrix. The equation then becomes

$$P_d = \mathrm{E}(\exp[\max_i E_i(\mathbf{z})]), \tag{6}$$

where $E_i(\mathbf{z})$ is Gaussian distributed with moments

$$\begin{gathered} \mathrm{E}(E_i(\mathbf{z})) = -\frac{1}{2}\lambda_{ii}\lambda^2 + \log p_i \\ \mathrm{Cov}(E_i(\mathbf{z}), E_j(\mathbf{z})) = \mathbf{s}_i^T\mathbf{s}_j = \lambda_{ij}\lambda^2. \end{gathered}$$

Note that the probability of detection in Eq. (6) is only a function of the SNR λ, the normalized matrix of inner products α, and the *a priori* signal probabilities p_i. This is because the Gaussian distribution is completely specified in terms of its first and second moments. Since α is invariant under orthogonal transformations imposed on the signal vectors, the probability of detection is invariant under any orthogonal transformation of the signal vectors. It is known that the inner product matrix or Gram matrix of the signal vectors is positive semidefinite [15, p. 407]. The following lemma is important to the forthcoming development:

Lemma 1 – *Every inner product matrix uniquely specifies a signal set-up to a geometric isometry.*

The proof is shown in Appendix A. This lemma shows that signal sets may be represented in terms of their inner product matrices. Hence, we define the admissible inner product matrices $\mathcal{A}$ to be those inner product matrices that are positive semidefinite and satisfy the average power constraint in Eq. (3). It is seen that $\mathcal{A}$ is a compact convex set. The boundary of $\mathcal{A}$ is the set of inner product matrices in $\mathcal{A}$ with determinant identically zero. The interior is the set of inner product matrices in $\mathcal{A}$ with determinant greater than zero. Since $\mathcal{A}$ is compact, and the probability of detection is a continuous function of α [10], it follows that the vector α, which maximizes the probability of detection, is found in $\mathcal{A}$. It is shown in Ref. [7, p. 63] that a necessary condition for optimality is that the sum of the signal vectors is identically zero. An equivalence of $\sum_i \mathbf{s}_i = 0$ is [7, p. 66]

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_{ij} = 0. \tag{7}$$

Another equivalence is

$$\sum_{i=1}^{M} \lambda_{ij} = 0, \quad j = 1, \cdots, M. \tag{8}$$

Equation (7) specifies a hyperplane that contains nonadmissible as well as admissible α. The intersection of the hyperplane with $\mathcal{A}$ contains the optimal α.

A partial ordering of $\mathcal{A}$ can be found as follows. Let two inner product matrices α and α' have the same diagonal elements. If $\lambda'_{ij} \leq \lambda_{ij}$ for all $i \neq j$ then $P_d(\lambda, \alpha') \geq P_d(\lambda, \alpha)$. This follows, since the derivative of the detection probability with respect to each λ_{ij} $i \neq j$ is nonpositive [7, p. 68].

Definition 1 — *$\{\mathbf{s}_1, \cdots, \mathbf{s}_M\}$ is a regular simplex if each $\mathbf{s}_i$, $i = 1, \cdots, M$ is at the same distance from each $\mathbf{s}_j$, $j = 1, \cdots, M$ where $i \neq j$. If additionally $\sum_i \mathbf{s}_i = \mathbf{0}$, the regular simplex is a regular simplex signal set.*

The regular simplex signal set can be specified in terms of the inner product matrix by the relations,

$$\begin{aligned} \lambda_{ii} &= 1, \qquad i = 1, \cdots, M \\ \lambda_{ij} &= -\frac{1}{M-1} \qquad i, j = 1, \cdots, M \quad i \neq j. \end{aligned} \tag{9}$$

The probability of detection of the simplex P_{d_s} is given by Ref. [11, p. 162]

$$P_{d_s} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-\frac{(x - \lambda\sqrt{\frac{M}{M-1}})^2}{2}] \Phi^{M-1}(x) dx \quad M \geq 2 \tag{10}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp[-\frac{x^2}{2}] dx. \tag{11}$$

A necessary condition for optimality at low SNR is that the mean width of the polytope generated by the signal set be a maximum. This is true for both the average power constrained case [6] and the peak power constrained case [5]. The polytope of a set of signal vectors $\{\mathbf{s}_i\}$ is the convex hull,

$$\{\mathbf{y} | \mathbf{y} = \sum_i \gamma_i \mathbf{s}_i, \quad \sum_i \gamma_i = 1, \quad \gamma_i \geq 0, \quad i = 1, \cdots, M\}.$$

The mean width B is defined as

$$B = \int_{\Omega_n} \max_i \mathbf{y}_n^T \mathbf{s}_i \, d\mu(\mathbf{y}_n), \tag{12}$$

where Ω_n is the surface of an n dimensional unit sphere, $\mathbf{y}_n$ is a unit vector representing a point on Ω_n, and $\mu(\mathbf{y}_n)$ is a uniform probability measure over Ω_n as a function of $\mathbf{y}_n$.

3. COUNTEREXAMPLE TO THE STRONG SIMPLEX CONJECTURE

An explicit signal set is presented that has a probability of detection greater than the regular simplex signal set. Consider placing $M-2$ signals at the origin, and placing the remaining two vectors in a manner such that they form an antipodal signal set as in Fig. 1. (Two equal length vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are defined to be antipodal whenever $\mathbf{s}_1^T \mathbf{s}_2 = -||\mathbf{s}_1|| \, ||\mathbf{s}_2||$ where $||\cdot||$ represents the Euclidean or L_2 norm) . We will refer to this signal set as the low SNR 1-D signal set ($L1$). Figure 1 shows the distance z from the origin to $\mathbf{s}_1$ or $\mathbf{s}_2$, which is determined by Eq. (2), thus yielding $z = \lambda\sqrt{\frac{M}{2}}$. When the signals are equally likely, it is easily seen from Eq. (4) that the optimal detector chooses the signal that is closest in Euclidean distance to the received signal. We define the decision regions Λ_i such that if $\mathbf{y} \in \Lambda_i$ we choose $\mathbf{s}_i$. The decision regions are shown in Fig. 1. If $\mathbf{y} \in \Lambda_3^M \equiv \{\Lambda_3, \cdots, \Lambda_M\}$ any $\mathbf{s}_i$, $i \in \{3, \cdots, M\}$ can be chosen. For simplicity, we will assume that if $\mathbf{y} \in \Lambda_3^M$ then $\mathbf{s}_3$ is chosen. Denoting P_{d_i} as the probability of detection of signal $\mathbf{s}_i$, we can compute the average probability of detection of the $L1$ signal set $P_{d_{L1}}$ as,

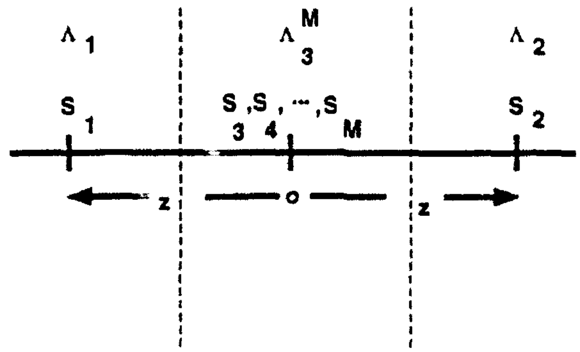

Fig. 1 — L1 Code

$$P_{d_{L1}} = \frac{1}{M}\sum_{i=1}^{M} P_{d_i}. \tag{13}$$

Now by symmetry $P_{d_1} = P_{d_2}$ and

$$\begin{aligned} P_{d_1} &= P(\mathbf{y} \in \Lambda_1 | \mathrm{s}_1) \\ &= \Phi(\frac{z}{2}) \end{aligned} \tag{14}$$

where $\Phi(x)$ is defined in Eq. (11). It is seen that $P_{d_4} = \cdots = P_{d_M} = 0$ and denoting $\delta \in \{\mathrm{s}_1, \mathrm{s}_2, \cdots, \mathrm{s}_M\}$ as the decision that is made regarding which signal is sent, we have

$$\begin{aligned} P_{d_3} &= P(\delta = \mathrm{s}_3 | \mathrm{s}_3) \\ &= P(\delta = \mathrm{s}_3 | \mathbf{y} \in \Lambda_3^M, \mathrm{s}_3) P(\mathbf{y} \in \Lambda_3^M | \mathrm{s}_3) + P(\delta = \mathrm{s}_3 | \mathbf{y} \in \overline{\Lambda_3^M}, \mathrm{s}_3) P(\overline{\Lambda_3^M} | \mathrm{s}_3) \\ &= 2\Phi(\frac{z}{2}) - 1 \quad (M \geq 3). \end{aligned} \tag{15}$$

Now the average probability of detection can be computed as

$$\begin{aligned} P_{d_{L1}} &= \frac{1}{M}[2P_{d_1} + P_{d_3}] \\ &= \frac{1}{M}[4\Phi(\frac{z}{2}) - 1] \\ &= \frac{1}{M}[4\Phi(\frac{\lambda}{2}\sqrt{\frac{M}{2}}) - 1] \quad (M \geq 3). \end{aligned} \tag{16}$$

Proposition 1 — *Neighborhoods $\lambda \in [0, \delta_M)$, $\delta_M > 0$ of λ, exist where, $P_{d_{L1}}(\lambda)$ is strictly greater than $P_{d_s}(\lambda)$ for all $M \geq 7$ and less than $P_{d_s}(\lambda)$ for $3 \leq M \leq 6$.*

Proof: Assume M is fixed. Since both $P_{d_{L1}} = P_{d_{L1}}(\lambda, M)$ and $P_{d_s} = P_{d_s}(\lambda, M)$ are differentiable with respect to λ on $\lambda \in [0, \infty)$ and $P_{d_{L1}} = P_{d_s} = 1 - \frac{1}{M}$ at $\lambda = 0$ it is sufficient [16, p. 209] to prove that

$$P'_{d_{L1}}(0, M) \equiv \left.\frac{\partial P_{d_{L1}}(\lambda, M)}{\partial \lambda}\right|_{\lambda=0} > \left.\frac{\partial P_{d_s}(\lambda, M)}{\partial \lambda}\right|_{\lambda=0}. \tag{17}$$

Now

$$\begin{aligned} P'_{d_{L1}}(0, M) &= \frac{4}{M} \left.\frac{\partial \Phi(\frac{\lambda}{2}\sqrt{\frac{M}{2}})}{\partial \lambda}\right|_{\lambda=0} \\ &= \frac{4}{M} \left.\frac{\partial \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\lambda}{2}\sqrt{\frac{M}{2}}} \exp[-\frac{x^2}{2}]dx}{\partial \lambda}\right|_{\lambda=0} \\ &= \frac{1}{\sqrt{M\pi}} \quad (M \geq 3). \end{aligned} \tag{18}$$

Since the integrand of Eq. (10) is differentiable with respect to λ on $\lambda \in [0, \infty)$, and the derivative

$$\sqrt{\frac{M}{M-1}}(x - \lambda\sqrt{\frac{M}{M-1}}) \exp[-\frac{(x - \lambda\sqrt{\frac{M}{M-1}})^2}{2}]\Phi^{M-1}(x)dx,$$

is integrable, it follows from Ref. [17, p.215] that the order of differentiation and integration can be reversed. Hence,

$$P'_s(0, M) \equiv \left.\frac{\partial P_{d_s}}{\partial \lambda}\right|_{\lambda=0} = \sqrt{\frac{M}{2\pi(M-1)}} \int_{-\infty}^{\infty} x \exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx. \tag{19}$$

Since the integrand in Eq. (19) is negative for $x < 0$ it follows that,

$$P'_s(0, M) < \sqrt{\frac{M}{2\pi(M-1)}} \int_{0}^{\infty} x \exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx. \tag{20}$$

For large M, the function Φ^{M-1} resembles a step function, with the location of the step increasing in M. The location of the step increases slowly with M, so we will approximate the location of the step by $c\log(M-1)$ and upper bound Eq. (20) by

$$P'_s(0, M) < \sqrt{\frac{M}{M-1}} \left[\int_0^{c\log(M-1)} c\log(M-1) \frac{\exp[-\frac{x^2}{2}]}{\sqrt{2\pi}} \Phi^{M-1}(x)dx + \int_{c\log(M-1)}^{\infty} x \frac{\exp[-\frac{x^2}{2}]}{\sqrt{2\pi}} dx \right]$$

where $c \geq 0$ is a constant to be optimized later, and log represents the natural logarithm. Both terms above can be directly integrated yielding,

$$
\begin{aligned}
P_s'(0,M) &< \sqrt{\frac{M}{M-1}}\left[\frac{c\log(M-1)}{M}(\Phi^M(c\log(M-1))-(\frac{1}{2})^M)+\frac{\exp[-\frac{(c\log(M-1))^2}{2}]}{\sqrt{2\pi}}\right] \\
&< \sqrt{\frac{M}{M-1}}\left[\frac{c\log(M-1)}{M}+\frac{\exp[\log((M-1)^{-\frac{1}{2}})\log((M-1)^{c^2})]}{\sqrt{2\pi}}\right] \\
&< \frac{c\log(M-1)}{\sqrt{M(M-1)}}+\sqrt{\frac{M}{2\pi(M-1)}}(\frac{1}{\sqrt{M-1}})^{c^2\log(M-1)}. \qquad (21)
\end{aligned}
$$

We want to determine when $P_{L1}' > P_s'$ or equivalently when $r \equiv P_s'/P_{L1}' < 1$. From Eq. (21) and Eq. (18)

$$
\begin{aligned}
r &< \frac{\frac{c\log(M-1)}{\sqrt{M(M-1)}}+\sqrt{\frac{M}{2\pi(M-1)}}(\frac{1}{\sqrt{M-1}})^{c^2\log(M-1)}}{\frac{1}{\sqrt{M\pi}}} \\
&= \sqrt{\pi}c\frac{\log(M-1)}{\sqrt{M-1}}+\frac{M}{\sqrt{2(M-1)}}(\frac{1}{\sqrt{M-1}})^{c^2\log(M-1)} \equiv h(M,c). \qquad (22)
\end{aligned}
$$

For each M define

$$c_M = \arg\min_c \quad h(M,c). \qquad (23)$$

We would like to determine c_M in order to find the smallest M such that $P_s'/P_{L1}' < 1$. By using an available minimization routine, we found $c_{30} = .78$ for $M = 30$ and $h(30,.78) = .9896$. To prove $r < 1$ for all $M \geq 30$ in Appendix B it is shown that $h(M,c)$ is a monotonically decreasing function of M for all $M \geq 30$.

The bound of Eq. (22) is not tight enough for the cases of $M \leq 30$. In Appendix C, a tighter upper bound is derived that is shown by Eq. (37) as

$$r < \sqrt{\frac{\pi}{(M-1)}}[-\frac{4}{k}\sum_{i=0}^{k-1}\Phi^M(i\frac{4}{k})+4\Phi^M(4)]+\frac{M}{\sqrt{2(M-1)}}\exp[-8], \qquad (24)$$

where the bound improves as the positive integer k increases. To evaluate rigorously an upper bound on r, we carry only five decimal places throughout any multiplication and round up or down the fifth decimal place as appropriate to upper bound r. We define t_1 as the evaluation in this manner of $\sum_{i=0}^{k-1}\Phi^M(i\frac{4}{k})$, and t_2 as the evaluation of $\Phi^M(4)$. We therefore upper bound Eq. (24) by

$$r < f_u(M) \equiv \sqrt{\frac{\pi}{(M-1)}}(-\frac{4}{k}t_1+4t_2)+\frac{M}{\sqrt{2(M-1)}}\exp[-8]. \qquad (25)$$

We have computed t_1 and t_2 by lower and upper bounding Φ functions for given k from a table of the Φ function to five decimal places [18]. The results can be verified and are shown in Table 1. This proves the result for $7 \leq M < \infty$.

Now we proceed to lower bound r for $M = 4, 5, 6$. The regular simplex was proven to be optimal for $M = 3$ [6] at low signal-to-noise ratios. In Appendix C, the tight lower bound of Eq. (38) is derived

$$r > -\frac{M}{\sqrt{2(M-1)}}\Phi^{(M-1)}(-1)\exp[-\frac{1}{2}] - \sqrt{\frac{\pi}{(M-1)}}\left[\sum_{i=0}^{19}(.05)\Phi^{M}(-.05i) + \Phi^{M}(-1)\right] +$$

$$\sqrt{\frac{\pi}{(M-1)}}\left[-\frac{4}{k}\sum_{i=1}^{k-1}\Phi^{M}(i\frac{4}{k}) + 4\frac{(k-1)}{k}\Phi^{M}(4)\right] + \frac{M}{\sqrt{2(M-1)}}\exp[-8]\Phi^{(M-1)}(4). \tag{26}$$

Again, we will carry five decimal places throughout the evaluation, rounding up or down appropriately to further lower bound r. We define q_1 as the evaluation in this manner of $-\frac{M}{\sqrt{2(M-1)}}\Phi^{(M-1)}(-1)\exp[-\frac{1}{2}] - \sqrt{\frac{\pi}{(M-1)}}\left[\sum_{i=0}^{19}(.05)\Phi^{M}(-.05i) + \Phi^{M}(-1)\right]$, q_2 as the evaluation of $\sum_{i=1}^{k-1}\Phi^{M}(i\frac{4}{k})$, $q_3 \leq \Phi^{M}(4)$, and q_4 as the evaluation of $\Phi^{(M-1)}(4)$. In Table 2 the lower bound is tabulated.

$$r > f_l(M) \equiv q_1 + \sqrt{\frac{\pi}{(M-1)}}\left[\frac{-4}{k}q_2 + 4\frac{k-1}{k}q_3\right] + \frac{M}{\sqrt{2(M-1)}}\exp[-8]q_4. \tag{27}$$

Again, the values of the Φ function were taken from Ref. [18]. In the computation of q_1, $\Phi(-1)$ and $\Phi(-.05i)$ the values were rounded up to the fifth decimal place, and $\Phi(-1)$ was rounded down. q_2 is the result of rounding up, and q_3 and q_4 are the results of rounding down. In the case of $M = 5$, k was found to be too large for the use of the table, so in this case an approximation from Ref. [19, Eq. 26.2.17] was used, which has an error $< 7.5 \cdot 10^{-8}$, and where we rounded the fifth decimal place up or down as before. Thus the proposition is proven.

This result, apart from being a counterexample to the SSC, leads to an important theorem.

Theorem 1 — *For any fixed $M \geq 7$ there is no signal set that is optimal at all signal-to-noise ratios.*

Proof: The proof follows from the proposition and Theorem 14 in Ref. [6] that established that the regular simplex is the only signal structure that is a local extremum at all $\lambda \geq 0$.

4. MAXIMIZATION OF MINIMUM DISTANCE

The normalized distance d_{ij}, between $\mathbf{s}_i$ and $\mathbf{s}_j$, is the Euclidean distance divided by λ. We define the normalized minimum distance d_{min}, as the minimum Euclidean distance divided by the SNR λ. Maximization of minimum Euclidean distance is equivalent to maximization of d_{min}. It was proven by both Balakrishnan [10] and Weber [11, p. 215] that the regular simplex uniquely maximizes the minimum distance under a peak power constraint $\lambda_{ii} = 1$, $i = 1, \cdots, M$. Here a new shortened proof of this is given. Note that maximizing the minimum distance for the peak power constrained channel is equivalent to minimizing $\max_{i \neq j} \lambda_{ij}$.

Theorem 2 — *The regular simplex is the unique signal set that maximizes the minimum distance between the signal vectors under a peak power constraint* $\lambda_{ii} = 1$, $i = 1, \cdots, M$.

Proof: The minimum value of $\max_{i \neq j} \lambda_{ij}$ is $-\frac{1}{M-1}$ since for any signal set α

$$\sum_i \mathbf{s}_i^T \sum_j \mathbf{s}_j \geq 0$$

implies that [11, p. 215],

$$\sum_i \sum_{j>i} \lambda_{ij} \geq -\frac{M}{2}. \tag{28}$$

Denoting the matrix of inner products of the regular simplex as α_s, any other matrix $\alpha' \neq \alpha_s$ that satisfies $\max_{i \neq j} \lambda'_{ij} = -\frac{1}{M-1}$ must have at least one λ_{ij} strictly less than $-\frac{1}{M-1}$, which violates Eq. (28). Therefore $\alpha' = \alpha_s$.

Now we will generalize this result to the case where the signal vectors are only constrained by $\sum_i \lambda_{ii} = M$. The difficulty found in solving this problem as opposed to the prior problem is that the distance between the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_j$ no longer depends only on the inner product λ_{ij}, $i \neq j$. The lemmas that follow will help to overcome this difficulty.

Lemma 2 — *A signal set with* $\sum_i \lambda_{ii} = M$ *maximizes the minimum distance* d_{min} *if and only if* $d_{ij}^2 = d_{min}^2 = \frac{2M}{M-1}$ *for all* $i \neq j$.

Proof: It can be seen that

$$\sum_i \sum_{j>i} d_{ij}^2 = \sum_i \sum_{j>i} \lambda_{ii} + \lambda_{jj} - 2\lambda_{ij} \tag{29}$$

$$= (M-1)\sum_{i=1}^{M}\lambda_{ii} - 2\sum_i\sum_{j>i}\lambda_{ij}$$

and by Eq. (28)

$$\sum_i\sum_{j>i} d_{ij}^2 \le M^2. \tag{30}$$

Since $\sum_i\sum_{j>i} 1 = \frac{M(M-1)}{2}$, we have $d_{min}^2\frac{M(M-1)}{2} \le M^2$ or $d_{min}^2 \le \frac{2M}{M-1}$, with equality if and only if $d_{ij} = d_{min}$ for all $i \neq j$. The latter condition is satisfied by the regular simplex signal set.

Now, we only need to prove that the regular simplex is the unique signal structure that maximizes d_{min}, since we know from above that it is a signal structure that maximizes d_{min}. Note that translations of a signal set result in the same distances between signal vectors.

Lemma 3 —*A necessary condition for maximizing the minimum distance is* $\sum_i \mathbf{s}_i = 0$.

Proof: Suppose $\sum_i \mathbf{s}_i \neq 0$ and assume (w.l.o.g.) $d_{min} > 0$. Consider the translation $f = -\frac{\sum_i \mathbf{s}_i}{M}$, $\mathbf{s}_i' = \mathbf{s}_i + f$ which implies $\sum_i \mathbf{s}_i' = 0$. By our earlier remark, $d_{min}' = d_{min}$. Now,

$$\begin{aligned}
\sum_i \lambda_{ii}\lambda^2 &= \sum_i (\mathbf{s}_i' - f)^T(\mathbf{s}_i' - f) \\
&= \sum_i \lambda_{ii}'\lambda^2 - 2\sum_i \mathbf{s}_i'^T f + f^T f \\
&= \sum_i \lambda_{ii}'\lambda^2 + f^T f.
\end{aligned} \tag{31}$$

Hence $\sum_i \lambda_{ii} = \gamma\sum_i \lambda_{ii}'$, where $\gamma > 1$. Since $\sum_i \lambda_{ii} = M$, we have $\sum_i \lambda_{ii}' = \frac{M}{\gamma}$, so we can rescale these signal vectors by forming another signal set $\mathbf{s}_i'' = \gamma\mathbf{s}_i$. Again note that $\sum_i \mathbf{s}_i'' = 0$, $\sum_i \lambda_{ii}'' = M$. The normalized distance between any two vectors of $\{\mathbf{s}_i''\}$ is

$$\begin{aligned}
d_{ij}''^2 &= \lambda_{ii}'' + \lambda_{jj}'' - 2\lambda_{ij} \\
&= \gamma(\lambda_{ii}' + \lambda_{jj}' - 2\lambda_{ij}) \\
&= \gamma d_{ij}'^2 \\
&= \gamma d_{ij}^2
\end{aligned}$$

where the last equality follows by our earlier remark. Since $\gamma > 1$, $d_{ij}'' \ge d_{ij}$ with equality if and only if $d_{ij} = 0$. Since we assumed $d_{min} > 0$, $d_{ij}'' > d_{ij}$ and hence $d_{min}'' > d_{min}$, which is a contradiction. This proves the necessary condition $\sum_i \mathbf{s}_i = 0$.

This leads to the theorem

Theorem 3 — *The regular simplex signal set is the unique signal set that maximizes the minimum distance.*

Proof: The result follows immediately from Lemmas 2 and 3 and from Definition 1.

This result leads to the corollary

Corollary 1 — *Given $\lambda > 0$, there exist signal sets that maximize the minimum distance between signals but do not maximize P_d.*

Proof: The result follows from Theorem 3, which states that the regular simplex is the unique signal set that maximizes d_{min}; and from Proposition 1, which demonstrates that the regular simplex is suboptimal in terms of P_d.

5. OPTIMALITY

In this section we address the optimality of signal sets in terms of both the average probability of error and the union bound on that probability. The union bound is an often used fairly tight approximation for low rates [20, p. 68] or relatively high SNRs when computing the average probability of error is intractable.

Theorem 4 — *The regular simplex uniquely minimizes the union bound.*

Proof: The union bound for the average probability of error of the ith signal, P_{e_i} is expressed as [20, p. 60]

$$P_{e_i} \leq \sum_{j:j \neq i} P_{e_{i \to j}} \tag{32}$$

where $P_{e_{i \to j}}$ is the error probability, given the ith signal is sent and the jth signal is the *only* alternative. The average probability of error is given by

$$P_e = \frac{1}{M} \sum_i P_{e_i}.$$

The union bound $P_e^{(u)}$ for the average probability of error is,

$$\begin{aligned} P_e \ \leq \ P_e^{(u)} &\equiv \frac{1}{M} \sum_i \sum_{j \neq i} P_{e_{i \to j}} \\ &= \frac{2}{M} \sum_i \sum_{j > i} P_{e_{i \to j}}. \end{aligned}$$

Now

$$P_{e_{ij}} = \Phi(-\frac{d_{ij}}{2}\lambda)$$

where d_{ij} is the normalized Euclidean distance between $\mathbf{s}_i$ and $\mathbf{s}_j$ as defined in Section 4. Hence,

$$P_e^{(u)} = \frac{2}{M}\sum_i\sum_{j>i}\Phi(-\frac{d_{ij}}{2}\lambda).$$

Let $\mathbf{u} = (d_{12}^2, d_{13}^2, \cdots, d_{(M-1)M}^2)$ where we include all d_{ij}, $i \neq j$. Let $f(u_i) = 2\Phi(-\frac{\sqrt{u_i}}{2}\lambda)$ where u_i is the ith element of $\mathbf{u}$. Hence $P_e^{(u)} = \frac{1}{M}\sum_{i=1}^{L} f(u_i)$ where $L = \frac{M(M-1)}{2}$ is the dimension of $\mathbf{u}$. Now $f(u_i)$ is a convex $\cup$ function on the interval $[0, \infty)$ since

$$f''(x) = \frac{\lambda}{2\exp[\frac{\lambda^2 x}{8}]\sqrt{8\pi}\,x^{\frac{3}{2}}} + \frac{\lambda^3}{\exp[\frac{\lambda^2 x}{8}]\sqrt{512\pi}\,\sqrt{x}} \geq 0,$$

for all $x \geq 0$. Hence Jensen's inequality applies:

$$\begin{aligned} P_e^{(u)} = \frac{1}{M}L\sum_{i=1}^{L}\frac{1}{L}f(u_i) &\geq \frac{M-1}{2}f(\sum_{i=1}^{L}\frac{1}{L}u_i) \\ &= \frac{M-1}{2}f(\frac{1}{L}\sum_i\sum_{j>i}d_{ij}^2). \qquad (33) \end{aligned}$$

The right-hand side of Eq. (33) is a strictly decreasing function of $\sum_i\sum_{j>i}d_{ij}^2$, hence it reaches its minimum value whenever $\sum_i\sum_{j>i}d_{ij}^2$ is maximum. We know from Lemma 3 that $\sum_i\sum_{j>i}d_{ij}^2 \leq M^2$ with equality if and only if $d_{min} = \frac{2M}{M-1}$. By Theorem 3 equality is achieved uniquely by the regular simplex, which is the only signal set that minimizes the right-hand side of Eq. (33). Note that for the case $d_{ij}^2 = \frac{2M}{M-1}$ Jensen's inequality is satisfied with equality as both sides become

$$\frac{M-1}{2}f(\frac{2M}{M-1}).$$

This completes the proof.

Corollary 2 — *There exists $\lambda > 0$ and signal sets $\{\mathbf{s}_i\}$ and $\{\mathbf{s}'_i\}$ such that $P_e^{(u)} < P_e^{(u')}$ whereas $P_e > P'_e$. The probabilities of error and union bounds for the signal sets $\{\mathbf{s}_i\}$ and $\{\mathbf{s}'_i\}$ are P_e, P'_e and $P_e^{(u)}$, $P_e^{(u')}$ respectively.*

Proof: Follows from proposition Eq. (1) and the previous theorem with $\{\mathbf{s}_i\}$ corresponding to the simplex.

We now will examine characteristics of optimal signal sets at low SNR.

Theorem 5 — *For all $M \geq 7$, the optimal solution at low SNR is not an equal energy solution.*

Proof: A necessary condition for optimality is that the mean width of the polytope Eq. (12) be maximized [6]. However, it was stated by Balakrishnan [5] that the regular simplex maximizes Eq. (12) among all equal energy signal sets. Since proposition Eq. (1) showed that the $L1$ signal set is better than the regular simplex for all $M \geq 7$, it is certainly better than all equal energy codes at low SNR. Hence, the optimal solution is not an equal energy solution.

Theorem 6 — *For sufficiently small λ, the $L1$ signal set is uniquely optimal in the class of 1-D signal sets that are average power constrained i.e., satisfy $\lambda^2 \leq P$.*

Proof: A necessary condition for optimality is again that the mean width of the polytope Eq. (12) be maximized. For signal sets restricted to one dimension this is equivalent to maximizing,

$$\begin{aligned} B_1 &\equiv \max_i s_i + \max_i -s_i, \\ &= \max_i s_i - \min_i s_i. \end{aligned} \tag{34}$$

Since detection probability can be shown to be an increasing function of λ [7, p. 62] the solution will lie on the boundary $\lambda^2 = P$. It has been remarked by Dunbridge and can also be seen by Eq. (12) that an optimal solution must have signal vectors either on the boundary of the convex hull generated by the signal vectors or else at the origin. Hence three possibilities are to be considered:

Case 1: $\{s_i\} = 0$ for $i = 1, \cdots, M - r$; and $\{s_j\} = x_1 > 0$ (or equivalently $\{s_j\}$ $x_1 < 0$) for $j = M - r + 1, \cdots, M$;

Case 2: $\{s_i\} = x_1 > 0$ and $\{s_j\} = x_2 > x_1$, (or equivalently $\{s_i\} = x_1 < 0$, $\{s_j\} = x_2 < x_1$) for $i = 1, \cdots, M - r$ and $j = M - r + 1, \cdots, M$;

Case 3: $\{s_i\} = 0$ for $i = 1, \cdots, M - r - p$; and $\{s_j\} = x_1 < 0$ for $j = M - r - p + 1, \cdots, M - p$; and $\{s_k\} = x_2 > 0$ for $k = M - p + 1, \cdots, M$.

A necessary condition for optimality [6] is that the convex hull generated by the signal vectors contain the origin. Hence, Case 2 is immediately dismissed. B_1 is seen to be maximized in Case 1 when $r = 1$ or $x_1 = \lambda\sqrt{M}$ resulting in $B_1 = \lambda\sqrt{M}$. However, the $L1$ code of Case 3 results in $B_1 = \lambda\sqrt{2M}$. Hence, Case 1 is dismissed. It can be seen, for any code in Case 3 with $r > 1$, that we can further decrease x_1 by rearranging $r - 1$ signals from x_1 to the origin. This will result in an increase in $B_1 = x_2 - x_1$. We can increase x_2 if $p > 1$. Hence, the optimal signal set belongs to Case 3 and is such that $r = p = 1$. Now, we only need to prove that $x_1 = -x_2$. This follows directly from the Cauchy-Schwartz inequality,

$$\sum_i r_i s_i \leq \sqrt{(\sum_i r_i^2)(\sum_i s_i^2)} \quad r_i, s_i > 0.$$

Let $s_1 = s_2 = 1$, $r_1 = x_1 > 0$, $r_2 = -x_2 > 0$ so that the above becomes,

$$\begin{aligned} x_1 - x_2 &\leq \sqrt{2(x_1^2 + x_2^2)} \\ &= \lambda\sqrt{2M}, \end{aligned}$$

where equality holds if and only if $(x_1, -x_2)^T$ is a multiple of to $(1, 1)$. This will occur if and only if $x_1 = -x_2 = \lambda\sqrt{\frac{M}{2}}$ proving optimality as well as uniqueness.

6. CONCLUSIONS

The problem of optimal signal design for the white Gaussian noise channel has been considered to be a fundamental problem for many years. We have exhibited a counterexample that disproves the long-standing strong-simplex conjecture. This counterexample essentially relies on the tradeoffs between communication rate and performance. When the communication rate is decreased by placing $M - 2$ signals at the origin, we can place the remaining two signals at a relatively large distance with the intent of at least discriminating between these two signals. At low signal-to-noise ratios, the performance is better than that of the regular simplex for which the distances between signal vectors is smaller than the distance between the two signals of the counterexample. This difference counteracts the effect that the high probability of error of the remaining $M - 2$ signals has on the performance of the $L1$ signal set.

This result in conjunction with previous work led to several theorems. It was established that the optimal signal set is indeed dependent on the signal-to-noise ratio for all $M \geq 7$. The optimal signal sets at low SNR for all $M \geq 7$ are necessarily unequal energy signal sets. The design of signal sets, under the criteria of maximizing the minimum distance, was also considered. Past results that established that the regular simplex maximizes the minimum distance under a peak power constraint were extended to the case of only an average power constraint. This led to the corollary that signal sets that maximize the minimum distance need not maximize the probability of detection. We also proved that the regular simplex optimizes the union bound. In general, optimization of the union bound does not necessarily indicate maximization of probability of detection. Finally the $L1$ signal set was shown to be the optimal signal set constrained to one dimension.

The determination of the optimal signal sets, which maximize the probability of detection, remains in general unsolved for low SNR. Perhaps optimal designs can be found for some partition of the SNR range $[0, \infty)$ as a function of M. The weak simplex conjecture has not been proven and the optimal design under a noncoherent assumption remain unsolved. With few exceptions, optimal signal designs under a bandwidth constraint remain unknown.

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REFERENCES

[1] T. M. Cover and B. Gopinath, *Open Problems in Communication and Computation.* Springer-Verlag, New York, N.Y. 1987.

[2] J. L. Massey, "Towards a proof of the simplex conjecture?," Unpublished Shannon Lecture at IEEE Int. Symp. Inform. Theory, June 1988.

[3] C. E. Shannon, "Probability of error of optimal codes in a gaussian channel," *B.S.T.J.*, **38**, 611–656 (May 1959).

[4] R. G. Gallager, *Information Theory and Reliable Communication.* John Wiley and Sons, New York, N.Y., 1968.

[5] A. V. Balakrishnan, "A contribution to the sphere-packing problem of communication theory," *Journal of Mathematical Analysis and Applications*, **3**, 485–506 (1961).

[6] B. Dunbridge, "Asymmetric signal design for the coherent Gaussian channel," *IEEE Trans. on Inform. Theory*, **IT-13**, 422–431, (1967).

[7] B. Dunbridge, Optimal signal design for the coherent Gaussian channel, Ph.D. thesis, Univ. of Southern California, L.A., Dec. 1965.

[8] H. J. Landau and D. Slepian, "On the optimality of the regular simplex code," *Bell Sys. Tech. J.*, **45**, 1247–1272, (Oct. 1966).

[9] S. M. Farber, On the signal selection problem for phase coherent and incoherent communication channels. Ph.D. thesis, California Institute of Technology, Pasadena, 1968.

[10] A. V. Balakrishnan, *Advances in communication systems* (McGraw-Hill, New York N.Y., 1965).

[11] C. L. Weber, *Elements of Detection and Signal Design*, (Springer Verlag, 1987).

[12] M. S. Stone and C. L. Weber, "On the globally optimum M-ary noncoherent digital communications systems," *Proc. IEEE*, **57**, 1203-1204, (1969).

[13] C. W. Helstrom, "The resolution of signals in white Gaussian noise," *Proc. IRE*, **43**, 1111–1118, (Jan. 1955).

[14] H. A. Krieger and C. A. Schaffner, "The global optimization of incoherent phase signals," *SIAM J. Appl. Math.*, **21**, 573–589, (Dec. 1971).

[15] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, MA. 1985).

[16] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis*, (John Wiley & Sons, New York, N.Y. 1982).

[17] P. Billingsley, *Probability and Measure*, (John Wiley & Sons, New York, N.Y. 1985).

[18] J. F. Kenney, *Mathematics of Statistics*, 3rd ed., (D. Van Nostrand Company, Inc., New York, N.Y. 1954).

[19] M. Abramowitz and I. E. Stegun, *Handbook of mathematical functions* 9nth ed. **AMS-55**, U.S. G.P.O., Washington, D.C. (1970).

[20] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding*, (McGraw-Hill, Inc., New York, N.Y. 1979).

[21] R. A. Horn, NRL, Private communication, 1992.

Appendix A

REPRESENTATION OF INNER PRODUCT MATRIX

We prove that an inner product matrix uniquely determines a signal set up to an isometry. This is proven for complex matrices; although, we are only interested in real inner product matrices. Let $\mathcal{C}_n$ denote the set of all n by n complex matrices.

Definition 2 — *Two signal sets $\mathcal{A} = \{\mathbf{s}_{1a} \cdots \mathbf{s}_{Ma}\}$ and $\mathcal{B} = \{\mathbf{s}_{1b} \cdots \mathbf{s}_{Mb}\}$ are isometric if there exists a bijective transformation $T : \mathcal{A} \to \mathcal{B}$, called an isometry, such that $d(T\mathbf{s}_{ia}, T\mathbf{s}_{ja}) = d(\mathbf{s}_{ia}, \mathbf{s}_{ja})$ for all i, j where $d(x, y)$ is the Euclidean distance from x to y.*

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two signal sets of M vectors. Let n be the maximum of the dimension of the vectors in $\mathcal{A}$ and $\mathcal{B}$. Let A be a matrix whose columns are composed of the signal vectors of the first signal set, and B the matrix whose columns are the signal vectors of the second signal set. Suppose that the matrix of inner products corresponding to A (i.e., A^*A where $*$ denotes the Hermitian transpose of A) is the same as the inner product matrix corresponding to B. Embed A and B into n by n matrices A' and B' by adding, if necessary, zeroes to the right and the bottom of each matrix. It follows from Ref. [15, p. 414] that there exists a unique positive semidefinite matrix $P = (A'^*A')^{\frac{1}{2}} \in \mathcal{C}_n$ and a unitary matrix $V \in \mathcal{C}_n$ such that $A' = VP$ [21]. Similarly, $B' = WP$ (since the inner product matrix A'^*A' is the same for both signal sets), and hence $A' = (VW^*)B'$. The matrix VW^* is unitary and unitary matrices are easily shown to be isometries. The additional zero columns can be removed from A' and B' and the resulting matrices are still related by the unitary matrix VW^*. Since the additional rows of zeroes are inconsequential, we conclude that two signal sets of M vectors are isometric if the matrices of inner products A^*A and B^*B are identical. Note that the converse is not necessarily true, since an inner product matrix of a signal set α depends on the ordering of the columns of A.

Appendix B

PROOF OF MONOTONICITY

We prove that Eq. (22) is monotonically decreasing for $M \geq 30$. Consider the first term in Eq. (22)

$$f_1(M) \equiv \sqrt{\pi} c \frac{\log(M-1)}{\sqrt{M-1}}.$$

It is easily seen that the derivative of f_1 with respect to M is,

$$f_1'(M) = \sqrt{\pi} \frac{2 - \log(M-1)}{2(M-1)^{\frac{3}{2}}}.$$

Hence $f_1'(n) < 0$ whenever $\log(M-1) > 2$ or $M > 6$. Since f_1 is a continuous function of M, the first term is a monotonically decreasing function of M for $M > 6$. Next consider the second term in Eq. (22)

$$\begin{aligned} f_2(M) &\equiv \frac{M}{\sqrt{2(M-1)}} \left(\frac{1}{\sqrt{(M-1)}}\right)^{c^2 \log(M-1)} \\ &= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{\sqrt{M-1}}\right)^{c^2 \log(M-1)+1} + (M-1)\left(\frac{1}{\sqrt{M-1}}\right)^{c^2 \log(M-1)+1}\right. \\ &= \frac{1}{\sqrt{2}}\left((M-1)^{\frac{-1-c^2 \log(M-1)}{2}} + (M-1)^{\frac{1-c^2 \log(M-1)}{2}}\right). \end{aligned} \tag{35}$$

To analyze $f_2(M)$ examine the function

$$g(M) = (M-1)^{\alpha(M)},$$

where $\alpha(M)$ is any differentiable function of M. The derivative of $g(M)$ is

$$g'(M) = \frac{\alpha(M)(M-1)^{\alpha(M)}}{(M-1)} + (M-1)^{\alpha(M)} \log(M-1)\alpha'(M).$$

A sufficient condition for $g'(M) < 0$ for any $M \geq 2$ is $\alpha(M) < 0$ and $\alpha'(M) < 0$. Therefore a sufficient condition for the first term of $f_2(M)$ to have a derivative that is negative is that $\frac{-1-c^2\log(M-1)}{2} < 0$ and $-\frac{c^2}{2(M-1)} < 0$. Both of these conditions are satisfied for $M \geq 2$. The sufficient conditions for the second term of $f_2(M)$ are $\frac{1-c^2\log(M-1)}{2} < 0$ and $-\frac{c^2}{2(M-1)} < 0$. The first condition is satisfied whenever $M \geq 7$ for $c_{30} = .78$ where c_M is defined in Eq. (23). The second condition is satisfied for $M \geq 2$. Thus, the second term is monotonically decreasing for $M \geq 7$. In conclusion the bound in Eq. (22) is monotonically decreasing for all $M \geq 30$.

Appendix C

DERIVATION OF BOUNDS

We first derive an upper bound of r. Normalizing Eq. (20) by dividing through by Eq. (18) we have,

$$r < M\sqrt{\frac{1}{2(M-1)}}\int_0^\infty x\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx. \tag{36}$$

This can be bounded by

$$\begin{aligned} r &< M\sqrt{\frac{1}{2(M-1)}}\left[\frac{4}{k}\int_0^{\frac{4}{k}}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \frac{8}{k}\int_{\frac{4}{k}}^{\frac{8}{k}}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \cdots\right. \\ &+ \left. 4\int_{\frac{4(k-1)}{k}}^{4}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \int_4^\infty x\exp[-\frac{x^2}{2}]dx\right]. \end{aligned}$$

This can be integrated yielding

$$\begin{aligned} r &< \sqrt{\frac{\pi}{M-1}}\left[\frac{4}{k}[\Phi^M(\frac{4}{k}) - \Phi^M(0)] + \frac{8}{k}[\Phi^M(\frac{8}{k}) - \Phi^M(\frac{4}{k})] + \cdots\right. \\ &+ \left. 4[\Phi^M(4) - \Phi^M(4\frac{k-1}{k})]\right] + \frac{M}{\sqrt{2(M-1)}}\exp[-8] \\ &= \sqrt{\frac{\pi}{M-1}}\left[-\frac{4}{k}\sum_{i=0}^{k-1}\Phi^M(i\frac{4}{k}) + 4\Phi^M(4)\right] + \frac{M}{\sqrt{2(M-1)}}\exp[-8]. \end{aligned} \tag{37}$$

We now derive a lower bound of r. Normalizing Eq. (19) by dividing through by Eq. (18) we have ,

$$r = \frac{M}{\sqrt{2(M-1)}}\int_{-\infty}^\infty x\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx.$$

This is lower bounded by,

$$\begin{aligned} r &> \frac{M}{\sqrt{2(M-1)}}\left[\int_{-\infty}^{-1} x\exp[-\frac{x^2}{2}]\Phi^{M-1}(-1)dx - .05\int_{-.05}^{0.}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx - \right. \\ & .1\int_{-.1}^{-.05}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx - \cdots - \int_{-1}^{-.95}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \\ & \left.\int_0^\infty x\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx\right]. \end{aligned}$$

This can be integrated as

$$\begin{aligned} r &> -\frac{M}{\sqrt{2(M-1)}}\Phi^{M-1}(-1)\exp[-\frac{1}{2}] - \sqrt{\pi}(M-1)[\sum_{i=0}^{19}(.05)\Phi^{M}(-.05i) + \Phi^{M}(-1)] + \\ &+ \frac{M}{\sqrt{2(M-1)}}\int_0^\infty x\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx. \end{aligned}$$

Upon substituting $q_1 > -\frac{M}{\sqrt{2(M-1)}}\Phi^{M-1}(-1)\exp[-\frac{1}{2}] - \sqrt{\frac{\pi}{(M-1)}}[\sum_{i=0}^{19}(.05)\Phi^{m}(-.05i) + \Phi^{M}(-1)]$ the above is,

$$r > q_1 + \frac{M}{\sqrt{(M-1)}}\int_0^\infty x\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx,$$

which can be bounded by,

$$\begin{aligned} r &> q_1 + \frac{M}{\sqrt{2(M-1)}}\left[0\int_0^{\frac{4}{k}}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \frac{4}{k}\int_{\frac{4}{k}}^{\frac{8}{k}}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \cdots \right. \\ &+ \left. \frac{4(k-1)}{k}\int_{\frac{4(k-1)}{k}}^{4}\exp[-\frac{x^2}{2}]\Phi^{M-1}(x)dx + \Phi^{M-1}(4)\int_4^\infty x\exp[-\frac{x^2}{2}]dx\right] \\ &= q_1 + \sqrt{\frac{\pi}{(M-1)}}\left[\frac{4}{k}(\Phi^M(\frac{8}{k}) - \Phi^M(\frac{4}{k})) + \frac{8}{k}(\Phi^M(\frac{12}{k}) - \Phi^M(\frac{8}{k})) + \cdots \right. \\ &+ \left. \frac{4(k-1)}{k}(\Phi^M(4) - \Phi^M(\frac{4(k-1)}{k}))\right] + \frac{M\exp[-8]\Phi^{M-1}(4)}{\sqrt{2(M-1)}}. \end{aligned}$$

This reduces to

$$f_l(M) \equiv q_1 + \sqrt{\frac{\pi}{(M-1)}}\left[-\frac{4}{k}\sum_{i=1}^{k-1}\Phi^M(i\frac{4}{k}) + \frac{4(k-1)}{k}\Phi^M(4)\right] + \frac{M\exp[-8]\Phi^{M-1}(4)}{\sqrt{2(M-1)}}. \tag{38}$$